Compactifications of moduli of points and lines in the projective plane

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Main goal & motivation

Projective duality identifies the moduli space $\mathbf{B}_n$ parametrizing configurations of n linearly general points in $\mathbb{P}^2$ with the moduli space $\mathbf{X}(3,n)$ parametrizing configurations of n linearly general lines in $(\mathbb{P}^2)^\vee$. When considering degenerations of such objects, it is interesting to compare the resulting compactifications. The problem of constructing a compactification of $\mathbf{B}_n$ parametrizing degenerate n-pointed central fibers of Mustafin joins was proposed by Gerritzen and Piwek [1]. In this work, we pursue this program and compare the resulting compactification of $\mathbf{B}_n$ with Kapranov's Chow quotient compactification $\overline{\mathbf{X}}(3,n)$ [2].

Kapranov's comp'n $\overline{\mathbf{X}}(r,n)$

- $G^0(r,n) := G(r,n) \cap T$, where $T \subseteq \mathbb{P}^{\binom{n}{r}-1}$ is the maximal torus.
- $V \in G^0(r,n) \implies \mathbb{P}(V) \subseteq \mathbb{P}^{n-1}$. The restriction of the n-coordinate hyperplanes of $\mathbb{P}^{n-1}$ gives n linearly general hyperplanes in $\mathbb{P}(V) \cong \mathbb{P}^{r-1}$.
- $\mathbf{X}(r,n) := G^0(r,n)/\mathbb{G}_m^{n-1}$ is the moduli space parametrizing configurations of n linearly general hyperplanes in $\mathbb{P}^{r-1}$.

Def (Kapranov): Chow quotient comp'n $\overline{\mathbf{X}}(r,n) := G(r,n)//\mathbb{G}_m^{n-1}$.

Thm (Kapranov): $\overline{\mathbf{X}}(2,n) \cong \overline{\mathrm{M}}_{0,n}$.

Thm (Haking–Keel–Tevelev): $\overline{\mathbf{X}}(r,n)$ carries a family of KSBA stable pairs.

Alexeev: Generalization of $\overline{\mathbf{X}}(r,n)$ paramet. *weighted hyperplane arrangements*.

Gerritzen–Piwek's comp'n $\overline{\mathbf{B}}_n$

- $\mathbf{U}_n \subseteq (\mathbb{P}^2)^n$ open subset parametrizing n-tuples of points in general linear position.
- $\mathbf{B}_n := \mathbf{U}_n/\mathrm{PGL}_3$.
- $\mathbf{B}_n \cong \mathbf{X}(3,n)$ (Gelfand-MacPherson corr.)
- Consider the embedding

$$\mathbf{B}_n \hookrightarrow \prod_{\substack{\text{Ordered}\\ \text{quintuples}\\ \text{in } \{1,\ldots,n\}}} \mathbb{P}^2,$$
$$[(p_1,\ldots,p_n)] \mapsto (\ldots, q_v, \ldots),$$

where q_v is the image of p_{v_5} under the linear map sending $p_{v_1}, p_{v_2}, p_{v_3}, p_{v_4}$ to

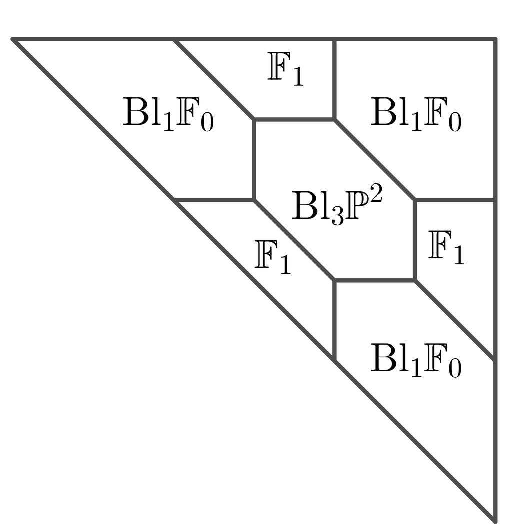
$$[1:0:0],\ [0:1:0],\ [0:0:1],\ [1:1:1].$$

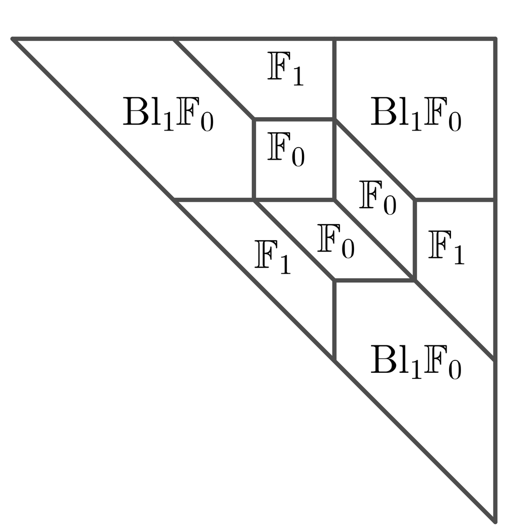
Def: $\overline{\mathbf{B}}_n :=$ Zariski closure of $\mathbf{B}_n \subseteq \prod \mathbb{P}^2$.

Rmk: Analogous construction for $\mathbb{P}^1$ yields $\overline{\mathbf{X}}(2,n)$ (HKT).

Q: $\overline{\mathbf{B}}_n \cong \overline{\mathbf{X}}(3,n)$? See Main Thm (i).

Rmk: $\overline{\mathbf{B}}_n$ constructed by Gerritzen–Piwek in relation to *Mustafin joins.*





Mustafin joins

- $R = \mathbb{C}[[t]]$, $K = Q(R)$, $\Bbbk = R/(t) \cong \mathbb{C}$.
- $\Sigma = \{L_1, \ldots, L_m\}$ free R-submodules of K^3 of rank 3.
- Define $\mathbb{P}(L_i) = \mathrm{Proj}(\mathrm{Sym}(L_i^\vee)) \cong \mathbb{P}^2_R$.
- Consider the natural embedding

$$\mathbb{P}^2_K \hookrightarrow \mathbb{P}(L_1) \times \cdots \times \mathbb{P}(L_m).$$

Def: The *Mustafin join* $\mathbb{P}(\Sigma)$ is the Zariski closure of $\mathbb{P}^2_K$ under the above embedding.

Def: Let $\mathbf{a} = (a_1, \ldots, a_n)\colon \mathrm{Spec}(K) \to \mathbf{B}_n$. A lattice L is *stable* provided $\exists$ 4 limits among $\overline{a}_1(0), \ldots, \overline{a}_n(0) \in \mathbb{P}(L)_\Bbbk \cong \mathbb{P}^2$ in g.l.p.

Def: $\Sigma_\mathbf{a} :=$ set of stable lattices w.r.t. $\mathbf{a}$ up to scaling by $K \setminus \{0\}$. Examples of central fibers $\mathbb{P}(\Sigma_\mathbf{a})_\Bbbk$ are in the left Figure.

Problem with $\overline{\mathbf{B}}_n$

Claim [1]: There exists $\pi\colon \overline{\mathcal{F}} \to \overline{\mathbf{B}}_n$ such that, for $x \in \overline{\mathbf{B}}_n$,

$$\pi^{-1}(x) \cong \mathbb{P}(\Sigma_\mathbf{a})_\Bbbk,$$

where $\mathbf{a}\colon \mathrm{Spec}(K) \to \mathbf{B}_n$ is an arc such that $\overline{\mathbf{a}}\colon \mathrm{Spec}(R) \to \overline{\mathbf{B}}_n$ satisfies $\overline{\mathbf{a}}(0) = x$.

Rmk (ST): $\exists$ $\mathbf{a}, \mathbf{b}\colon \mathrm{Spec}(K) \to \mathbf{B}_n$ such that $\overline{\mathbf{a}}(0) = \overline{\mathbf{b}}(0) \in \overline{\mathbf{B}}_6$ and $\mathbb{P}(\Sigma_\mathbf{a})_\Bbbk \not\cong \mathbb{P}(\Sigma_\mathbf{b})_\Bbbk$ (see the left Figure).

Main theorem (ST, 2020, [3])

(i) Gerritzen–Piwek's $\overline{\mathbf{B}}_n$ and Kapranov's $\overline{\mathbf{X}}(3,n)$ have isomorphic normalizations.

(ii) There exists a compactification $\mathbf{B}_n \subseteq \overline{\mathbf{X}}_{\mathrm{GP}}(3,n)$ with a proper flat family such that the fiber over $x \in \overline{\mathbf{X}}_{\mathrm{GP}}(3,n)$ is $\mathbb{P}(\Sigma_\mathbf{a})_\Bbbk$, where $\mathbf{a}\colon \mathrm{Spec}(K) \to \mathbf{B}_n$ is an arc such that $\overline{\mathbf{a}}(0) = x$.

(iii) $\overline{\mathbf{X}}_{\mathrm{GP}}(3,5) \cong \overline{\mathrm{M}}_{0,5}$ and $\overline{\mathbf{X}}_{\mathrm{GP}}(3,6)^\nu$ is a tropical compactification.

Alternative comp'n of $\mathbf{B}_n$

$\mathbf{H} :=$ *multigraded Hilbert scheme* of $(\mathbb{P}^2)^{\binom{n}{4}}$.

Def: $\overline{\mathbf{X}}_{\mathrm{GP}}(3,n) :=$ closure of $\mathbf{B}_n \hookrightarrow \overline{\mathbf{B}}_n \times \mathbf{H}$. $\overline{\mathcal{M}} \to \overline{\mathbf{X}}_{\mathrm{GP}}(3,n)$ pullback of the family over the regular Hilbert scheme.

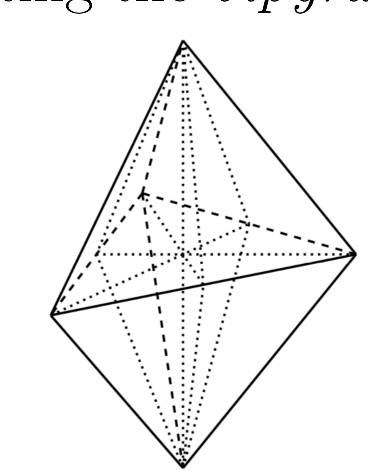
Prop (ST): If $\mathbf{a}\colon \mathrm{Spec}(K) \to \mathbf{B}_n$ and $\overline{\mathbf{a}}\colon \mathrm{Spec}(R) \to \overline{\mathbf{X}}_{\mathrm{GP}}(3,n)$, then $\overline{\mathbf{a}}^*\overline{\mathcal{M}} \cong \mathbb{P}(\Sigma_\mathbf{a})$. Moreover, $\overline{\mathbf{X}}_{\mathrm{GP}}(3,n) \xrightarrow{\text{birat.}} \overline{\mathbf{B}}_n$.

$\overline{\mathbf{X}}_{\mathrm{GP}}(3,6)^\nu$ is tropical

$\mathbf{X}(3,6) \subseteq \mathbb{G}_m^{\binom{6}{3}-1}/\mathbb{G}_m^5 \subseteq Y_{\Sigma(3,6)}$, where $\Sigma(3,6)$ is Speyer–Sturmfels' tropical Grassmannian.

Thm (Luxton): $\overline{\mathbf{X}}(3,6) \cong$ Zariski closure of $\mathbf{X}(3,6) \subseteq Y_{\Sigma(3,6)}$.

Thm (ST): $\overline{\mathbf{X}}_{\mathrm{GP}}(3,6)^\nu \cong$ Zariski closure of $\mathbf{X}(3,6) \subseteq Y_{\widehat{\Sigma}(3,6)}$ for $\widehat{\Sigma}(3,6) \preccurlyeq \Sigma(3,6)$ obtained by splitting the *bipyramid cones*:



References

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[2] M.M. Kapranov. *Chow quotients of Grassmannians. I.* I. M. Gel'fand Seminar, 29–110, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993.

[3] L. Schaffler and J. Tevelev. *Compactifications of moduli of points and lines in the projective plane.* Submitted. arXiv:2010.03519

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