On the cone of effective 2-cycles on $\overline{M}_{0,7}$

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Main Goal

Motivated by Fulton's question for effective k-cycles on $\overline{M}_{0,n}$, $1 < k < n-4$, this research aims to describe the cone $\mathrm{Eff}_2(\overline{M}_{0,7})$. Our main result is the construction of the subcone $V_2^{KV+CT}(\overline{M}_{0,7})$ obtained by enlarging in two steps the cone generated by the boundary 2-strata on $\overline{M}_{0,7}$.

Background & Motivation

- For $n \geq 3$, $\overline{M}_{0,n}$ is a $(n-3)$-dimensional smooth, projective, connected fine moduli space parametrizing stable n-pointed rational curves.
- $\mathrm{Eff}_k(\overline{M}_{0,n}) = \{\Sigma r_i Z_i \mid r_i \in \mathbb{R}_{\geq 0}, Z_i \subseteq \overline{M}_{0,n}, \dim(Z_i) = k\}/_{\text{"num. equivalence"}}$
- The ***boundary k-strata*** are the k-dimensional irreducible components of the boundary of $\overline{M}_{0,n}$.
- Denote by $V_k(\overline{M}_{0,n})$ the subcone of $\mathrm{Eff}_k(\overline{M}_{0,n})$ generated by the equivalence classes of the boundary k-strata on $\overline{M}_{0,n}$.

Fulton's Question

$V_k(\overline{M}_{0,n}) = \mathrm{Eff}_k(\overline{M}_{0,n})$?

- True for $k = 1$, $n = 5, 6, 7$ (see [3]).
- Open for $k = 1$, $n \geq 8$ (F-conjecture).
- False in codimension 1, $n \geq 6$: Keel and Vermeire gave examples of effective divisors on $\overline{M}_{0,n}$ which do not lie in $V_2(\overline{M}_{0,n})$ (see [4] and [6]).
- What about $1 < k < n-4$?

Lift of Effective Cycles

Lifting Lemma (S., 2015, [5]): Let $\{a, b\} \subset \{1, \ldots, n+1\}$ and consider the inclusion $\iota \colon \overline{M}_{0,n} \equiv D_{ab} \hookrightarrow \overline{M}_{0,n+1}$. Let $\alpha \in \mathrm{Eff}_k(\overline{M}_{0,n})$. The following hold:

- if $\alpha \notin V_k(\overline{M}_{0,n})$, then $\iota_* \alpha \notin V_k(\overline{M}_{0,n+1})$;
- if α is extremal in $\mathrm{Eff}_k(\overline{M}_{0,n})$, then $\iota_* \alpha$ is extremal in $\mathrm{Eff}_k(\overline{M}_{0,n+1})$.

Corollary: Given $1 < k < n-4$, then

$$V_k(\overline{M}_{0,n}) \subsetneq \mathrm{Eff}_k(\overline{M}_{0,7}).$$

Proof of the Corollary: Lift to $\overline{M}_{0,n}$ the Keel-Vermeire divisors on $\overline{M}_{0,k+1}$.

- New task: describe $\mathrm{Eff}_k(\overline{M}_{0,7}) \setminus V_2(\overline{M}_{0,7})$. The first interesting case is $n = 7$, $k = 2$.

The Cone $V_2(\overline{M}_{0,7})$

- $\sigma_{I,J,K}$ denotes the equivalence class of a boundary 2-stratum on $\overline{M}_{0,7}$, where $I \amalg J \amalg K = \{1, \ldots, 7\}$, $|I|, |K| \geq 2$ and $1 \leq |K| \leq 3$ (see Figure 1).
- The $\sigma_{I,J,K}$ generate 420 distinct rays of $\mathrm{Eff}_2(\overline{M}_{0,7})$. These rays are extremal by [2].

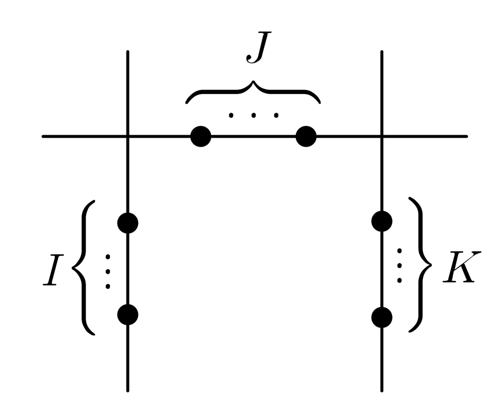

Figure 1: Stable 7-pointed rational curve parametrized by the generic point of a boundary 2-stratum.

Special Hypertree Surfaces

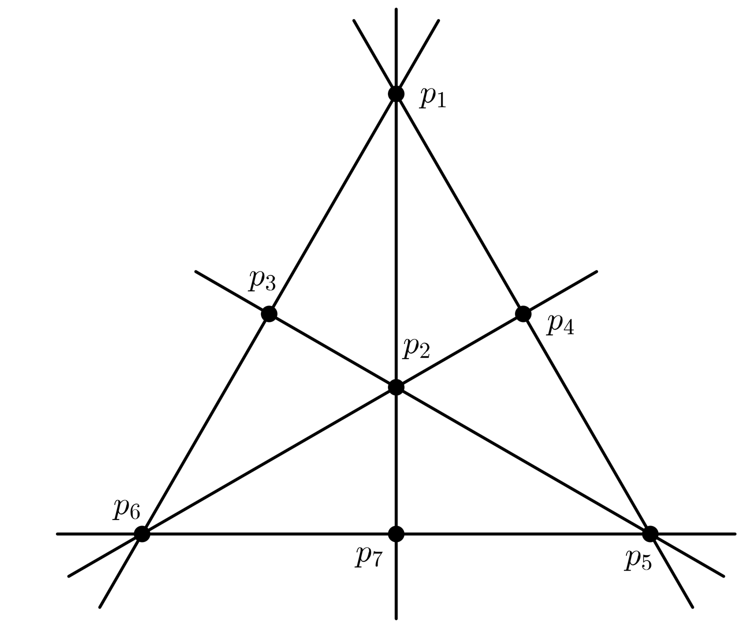

Figure 2: 7-points arrangement in $\mathbb{P}^2$ which gives a special hypertree surface $\mathrm{Bl}_{p_1,\ldots,p_7}(\mathbb{P}^2) \hookrightarrow \overline{M}_{0,7}$.

Further Directions

- $V_2^{KV+CT}(\overline{M}_{0,7}) = \mathrm{Eff}_2(\overline{M}_{0,7})$?
- What can we say about $\mathrm{Eff}_2(\overline{M}_{0,n}/S_n)$?

The Cone $V_2^{KV}(\overline{M}_{0,7})$

- $V_2^{KV}(\overline{M}_{0,7})$ is the cone generated by $V_2(\overline{M}_{0,7})$ and by the lifts of the Keel-Vermeire divisors on $\overline{M}_{0,6}$, which can be written as

$$\sigma_{im,jk\ell,ab} + \sigma_{jm,ik\ell,ab} + \sigma_{ij\ell m,k,ab} + \sigma_{ijkm,\ell,ab} + 2\sigma_{ijm,k\ell,ab} - \sigma_{ijk\ell,m,ab},$$

where $\{a, b, i, j, k, \ell, m\} = \{1, \ldots, 7\}$.

- There are 315 lifts generating distinct rays. By the Lifting Lemma, these lifts are extremal in $\mathrm{Eff}_2(\overline{M}_{0,7})$ and they lie outside of $V_2(\overline{M}_{0,7})$. So $V_2(\overline{M}_{0,7}) \subsetneq V_2^{KV}(\overline{M}_{0,7})$.

The Cone $V_2^{KV+CT}(\overline{M}_{0,7})$

- Given $p_1, \ldots, p_7 \in \mathbb{P}^2$ that do not lie on a (possibly reducible) conic, we have an embedding $\mathrm{Bl}_{p_1,\ldots,p_7}\mathbb{P}^2 \hookrightarrow \overline{M}_{0,7}$ (see [1]).
- $V_2^{KV+CT}(\overline{M}_{0,7})$ is the cone generated by $V_2^{KV}(\overline{M}_{0,7})$ and by these embedded blow ups of $\mathbb{P}^2$.
- To show $V_2^{KV}(\overline{M}_{0,7}) \subsetneq V_2^{KV+CT}(\overline{M}_{0,7})$ we consider the surface obtained by choosing the configuration in Figure 2.

References

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Main Theorem (S., 2015, [5])

$$V_2(\overline{M}_{0,7}) \subsetneq V_2^{KV}(\overline{M}_{0,7}) \subsetneq V_2^{KV+CT}(\overline{M}_{0,7}) \subseteq \mathrm{Eff}_2(\overline{M}_{0,7}).$$

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