The KSBA compactification of the moduli space of $D_{1,6}$-polarized Enriques surfaces

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Main Goal & Motivation

The aim of this research is to find a functorial and combinatorial compactification of the moduli space of Enriques surfaces. Model examples in the case of curves and abelian varieties are $\overline{M}_{g,n}$ and $\overline{A}_g^{\tau}$, where τ is the 2nd Voronoi fan. Compactifications of the moduli space of polarized K3 surfaces received a lot of attention ([2, 3]), which motivates us to consider the case of Enriques surfaces. Here we analyze $D_{1,6}$-polarized Enriques surfaces.

Background

Def: $D_{1,6} \subset \mathbb{Z} \oplus \mathbb{Z}^6(-1)$ is the sublattice of vectors of even square.

Def: An *Enriques surface* S is a smooth projective surface such that $2K_S \sim 0$ and $h^0(\omega_S) = h^1(\mathcal{O}_S) = 0$. A $D_{1,6}$-polarization on S is a primitive embedding $D_{1,6} \subset \mathrm{Pic}(S)$ with additional requirements.

Fact: A $D_{1,6}$-polarized Enriques surface can be realized as the $\mathbb{Z}_2^2$-cover of $\mathrm{Bl}_3\mathbb{P}^2$ branched along the configuration of lines L in Figure 1.

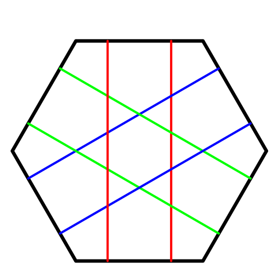

Figure 1: Three pairs of lines L on $\mathrm{Bl}_3\mathbb{P}^2$ (toric picture)

Def: A pair (B, D) where B is a variety and D a $\mathbb{Q}$-divisor on B is *stable* if

- $K_B + D$ is ample;
- (B, D) is *semi-log canonical*.

KSBA Compactification

Consider stable pairs $(S, \epsilon R)$: S is a $D_{1,6}$-polarized Enriques surface, $0 < \epsilon \ll 1$ and R is the ramification divisor of $S \to \mathrm{Bl}_3\mathbb{P}^2$.

- $\overline{M}^{\nu}_{D_{1,6}}$ = normalization of the projective coarse moduli space parametrizing stable pairs $(S, \epsilon R)$ and their degenerations (KSBA compactification).
- $\partial\overline{M}^{\nu}_{D_{1,6}}$ = boundary of $\overline{M}^{\nu}_{D_{1,6}}$ (i.e. the set of points parametrizing reducible pairs).

Goal: Describe $\partial\overline{M}^{\nu}_{D_{1,6}}$ and the degenerations parametrized by it. Relate $\overline{M}^{\nu}_{D_{1,6}}$ with the corresponding Baily-Borel compactification.

Reduction: Equivalent to study stable pairs $\left(\mathrm{Bl}_3\mathbb{P}^2, \left(\frac{1+\epsilon}{2}\right) L\right)$.

Baily-Borel Compactification

- $\overline{\mathcal{D}/\Gamma}^{BB}$ = Baily-Borel compactification of the period domain $\mathcal{D}$ for $D_{1,6}$-polarized Enriques surfaces (studied in [4]).
- The boundary $\partial\overline{\mathcal{D}/\Gamma}^{BB}$ consists of two rational 1-cusps and three 0-cusps.

Stratification of $\partial\overline{M}^{\nu}_{D_{1,6}}$

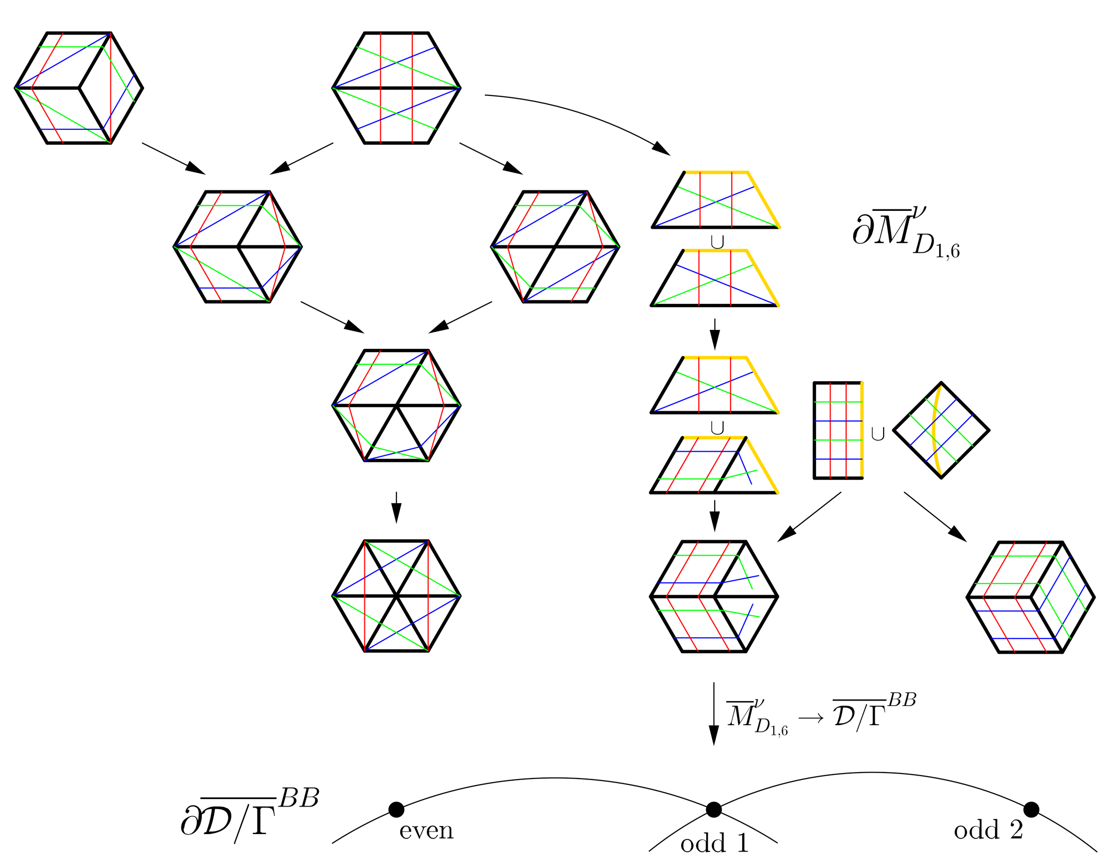

Figure 2: Comparison between the boundaries of the KSBA and Baily-Borel compactifications

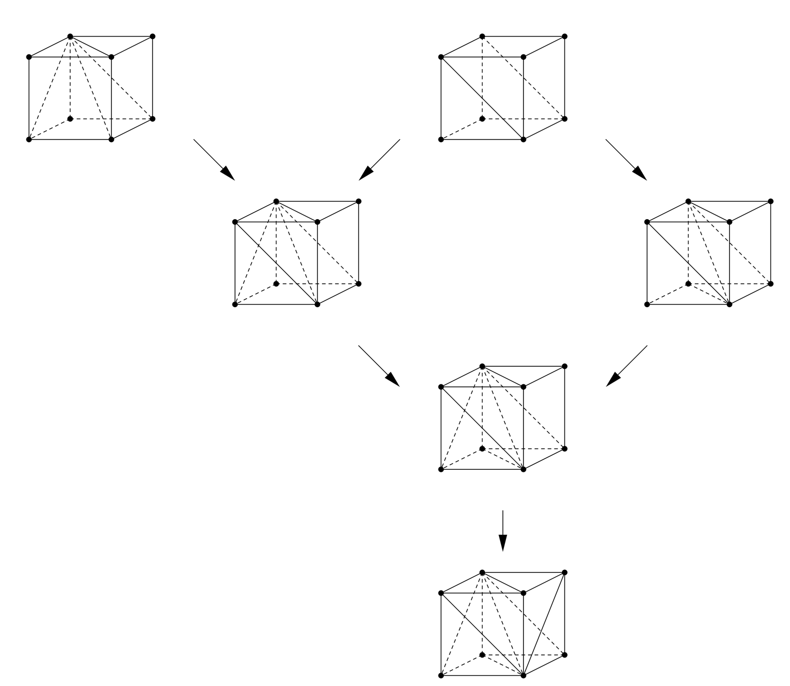

Figure 3: Combinatorial interpretation of part of $\partial\overline{M}^{\nu}_{D_{1,6}}$

Main Theorem (S., 2016, [5])

(i) $\partial\overline{M}^{\nu}_{D_{1,6}}$ has two divisorial components and one of codimension 3. For the complete stratification of the boundary and the degenerations parametrized by it, see Figure 2.

(ii) There exists a birational map $\overline{M}^{\nu}_{D_{1,6}} \to \overline{\mathcal{D}/\Gamma}^{BB}$. In a neighborhood of the preimage of the even (resp. odd 2) 0-cusp, $\overline{M}^{\nu}_{D_{1,6}}$ has a toroidal behavior (resp. is isomorphic to Baily-Borel). Otherwise, it is a mixture of toroidal and Baily-Borel. The combinatorics of the toroidal behavior is interpreted in terms of specific polyhedral subdivisions of the unit cube (Figure 3), or in terms of elliptic and maximal parabolic subdiagrams of a certain Coxeter diagram.

Main Ideas for the Proof

- $\left(\mathrm{Bl}_3\mathbb{P}^2, \left(\frac{1+\epsilon}{2}\right) L\right) \cong \left(B, \left(\frac{1+\epsilon}{2}\right) \Delta|_B\right)$: $B \subset (\mathbb{P}^1)^3$ divisor of class $(1,1,1)$, $\Delta \subset (\mathbb{P}^1)^3$ toric boundary. $((\mathbb{P}^1)^3, B)$ is a *stable toric pair of type* $\leq Q = [0,1]^3$.
- $\overline{M}^{\nu}_Q$ = normalization of the projective coarse moduli space parametrizing stable toric pairs of type $\leq Q = [0,1]^3$.
- $\overline{M}^{\nu}_Q \to \overline{M}^{\nu}_{D_{1,6}}$ generically given by $(X, B) \mapsto \left(B, \left(\frac{1+\epsilon}{2}\right) \Delta|_B\right)$. This describes $\partial\overline{M}^{\nu}_{D_{1,6}}$. (We also show that $\overline{M}^{\nu}_{D_{1,6}} \cong \overline{M}^{\nu}_Q/\mathrm{Sym}(Q)$.)
- To extend $\overline{M}^{\nu}_{D_{1,6}} \dashrightarrow \overline{\mathcal{D}/\Gamma}^{BB}$ we use Kulikov semistable degenerations of K3 surfaces.

Future Project

I am extending the results to a 10-dimensional family of polarized Enriques surfaces.

References

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